Lecture 9

In this lecture we are going to prove one of the most important theorems not only in group theory but in all of mathematics - Lagrange's theorem.

For that we'll introduce the notion of coset and use it's properties to prove Lagrange's theorem and then many of it's important corollaries.

Definition Let G be a group and let H be a subgroup of G. For any $a \in G$, the set

$$aH = \{ah \mid h \in H\}$$

is called the left coset of H in G containing a and the set

$Ha = \{ha \mid h \in H\}$ is called the right coset of H in G containing a.

Definition Let G be a group and $H \leq G$. The index of H in G, denoted $|G:H|$, is the number of left (or right) cosets of H in G.

The definitions given above are not very illu-minating in the sense that they do not give us a "feel" of what these objects.

Why are we defining both right and left cosets?

Are these also subgroups of G?

What is the role of $a \in G$ in the definition?

To answer all these questions, let's see some examples.

Examples

1. Let $G = (\mathbb{Z}_9, +)$ and $H = \{0, 3, 6\}$. Since the group operation is $+$, so we use $a + H$ instead of $a \cdot H$. Let's find $a + H$ for all $a \in \mathbb{Z}_9$.

$$0 + H = \{0 + h \mid h \in H\} = \{0, 3, 6\} = 3 + H = 6 + H$$

$$1 + H = \{1 + h \mid h \in H\} = \{1, 4, 7\} = 4 + H = 7 + H$$

$$2 + H = \{2, 5, 8\} = 5 + H = 8 + H$$

2. Let $G = S_3$ and $H = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \right\}$.

Let $a = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$. What is aH and Ha?

$$aH = \{ah \mid h \in H\} = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \right\}$$

$$Ha = \{ha \mid h \in H\} = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \right\}$$

Let's see these examples carefully!

The first example tells us many things :-

* $a+H$ might not be a subgroup of G as $1+H$ is not a subgroup of $\mathbb{Z}_9$.
* $a+H$ can be equal to $b+H$ even if $a \neq b$ as $1+H = 4+H = 7+H$
* $a+H = H$ when $a \in H$ in this example.

The second example tells us that why is it important to define both left and right cosets as clearly in the second example $aH \neq Ha$.

Now in the definition of the index of H in G, we said the number of either left or right

cosets. This requires justification, i.e., the
(right cosets of H in G) = # (right cosets of H in G)
as is an exercise.

We want to use cosets to prove

<u>Theorem [Lagrange's Theorem]</u> Let G be a finite group and $H < G$ a subgroup. Then $|H|$ divides $|G|$.

How are we going to use cosets to prove Lagrange's theorem? First, we will work with left cosets, however, nothing will change if you work with right cosets. The strategy is then as follows:- <u>Strategy</u>

1) Show that the left cosets partition the group, i.e., every group element occurs in some coset and occurs only in that coset and none other.

2) Show that the size of all the cosets are the same.

The following lemma not only proves both the points in the strategy but many more properties of cosets.

Lemma Let G be a group and $H < G$. Let $a, b \in G$. Then

1) $a \in aH$

2) $aH = H$ if and only if $a \in H$.

3) $aH = bH$ if and only if $a \in bH$.

4) $aH = bH$ or $aH \cap bH = \phi$

5) $aH = bH$ if and only if $a^{-1}b \in H$.

6) $|aH| = |bH|$.

7) $aH = Ha$ if and only if $aHa^{-1} = H$.

8) aH is a subgroup of G if and only if $a \in H$.

Remark Note that Property (1) in the Lemma is telling us that every element of G is in some coset. Property (4) is telling us that an element can belong to only one coset and Property (6) is telling us that all the cosets have the same size and we even know what that would be, it would be $|H|$.

Proof:- Properties 2), 7) and 8) are left as an exercise. Let's prove all the others.

1) $a = a.e$ and $e \in H \Rightarrow a \in aH$.

3) $\Leftarrow$ If $a \in bH \Rightarrow a = bh$ for some $h \in H$.

Let ah_1 be an arbitrary element of aH.

$\Rightarrow ah_1 = bh.h_1 = b.(hh_1) \in bH \Rightarrow aH \subseteq bH$.

Similarly, one can prove $bH \subseteq aH \Rightarrow aH = bH$.

$\Rightarrow$ If $aH = bH \Rightarrow$ the sets

$$\{ah \mid h \in H\} = \{bh \mid h \in H\}$$

$\Rightarrow$ $ah_1 = bh_2$ for some $h_1, h_2 \in H$

$\Rightarrow$ $a = bh_2h_1^{-1} \in bH$.

4) If $aH \cap bH = c$ $\Rightarrow$ $c \in aH \Rightarrow cH = aH$ from property 3) and $c \in bH \Rightarrow cH = bH$

$\Rightarrow$ $aH = bH$, otherwise $aH \cap bH = \phi$.

5) $\Rightarrow$ Let $aH = bH$. Then as in the proof of 3), $ah_1 = bh_2$ for some $h_1, h_2 \in H$.

$\Rightarrow$ $h_1 = a^{-1}bh_2$ $\Rightarrow$ $h_1h_2^{-1} = a^{-1}b$ $\Rightarrow$

$a^{-1}b \in H$.

$\Leftarrow$ Let $a^{-1}b \in H$ $\Rightarrow$ from property 1)

$a^{-1}bH = H$ $\Rightarrow$ $a(a^{-1}bH) = aH \Rightarrow bH = aH$.

6) Since aH and bH are sets, if we want

to show that they have equal number of elements, it's enough to find a bijection between them. Define

$$\alpha : aH \longrightarrow bH \quad \text{by} \quad \alpha(ah) = bh$$

Clearly α is onto. For one-one, suppose $\alpha(ah_1) = \alpha(ah_2) \Rightarrow bh_1 = bh_2 \Rightarrow h_1 = h_2$ by cancellation $\Rightarrow ah_1 = ah_2$. Thus α is a bijection and so $|aH| = |bH|$.

Using this Lemma, we'll prove the Lagrange theorem in the next lecture.

<u>Exercise</u> Try to understand the properties of cosets and their proofs clearly.

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